Quasi normal modes

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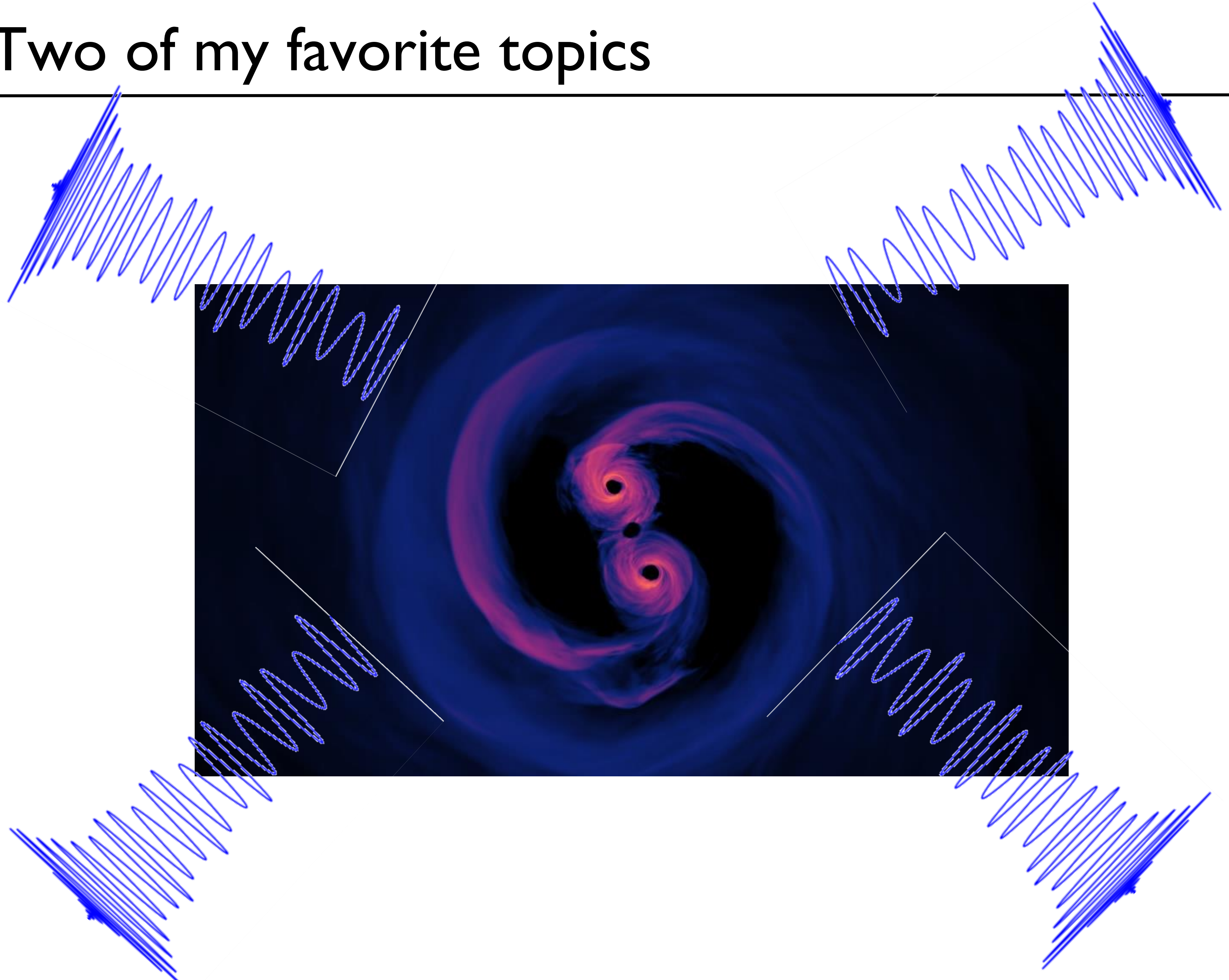
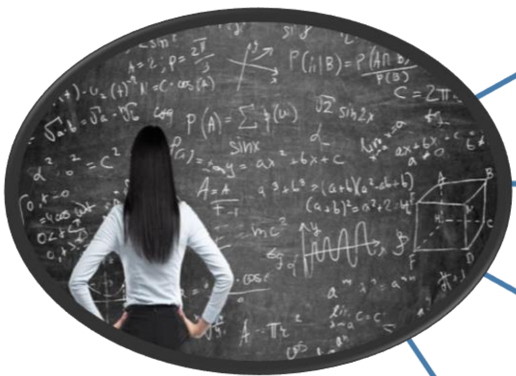
Two of my favorite topics



Various expansion parameters



Post-Minkowskian

- Background: Minkowski
- Parameter: G
- Application: Scattering amplitudes, EFT, often combined with post-Newtonian expansions

Post-Newtonian

- Background: Minkowski
- Parameter: $1/c^2$
- Application: GWs during inspiral

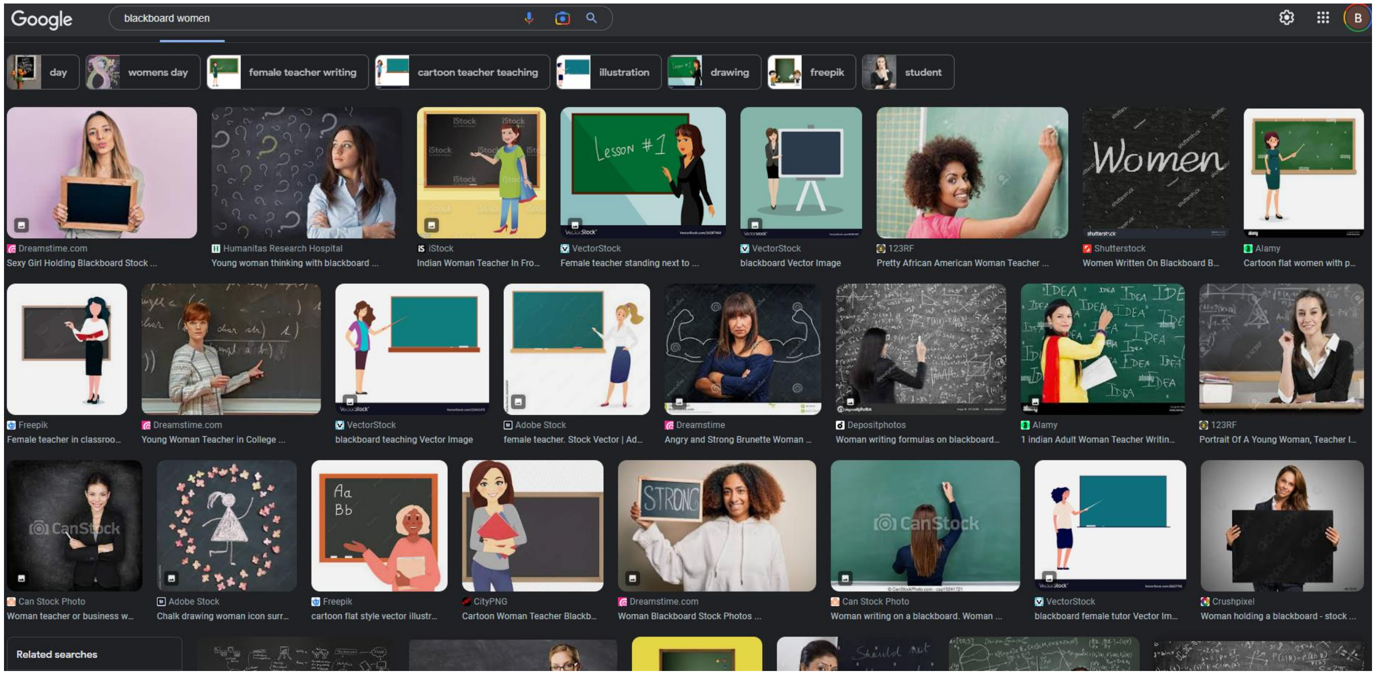
Schwarzschild perturbation theory

- Background: Schwarzschild
- Parameter: often perturbing mass m
- Application: EMRIs, QNMs

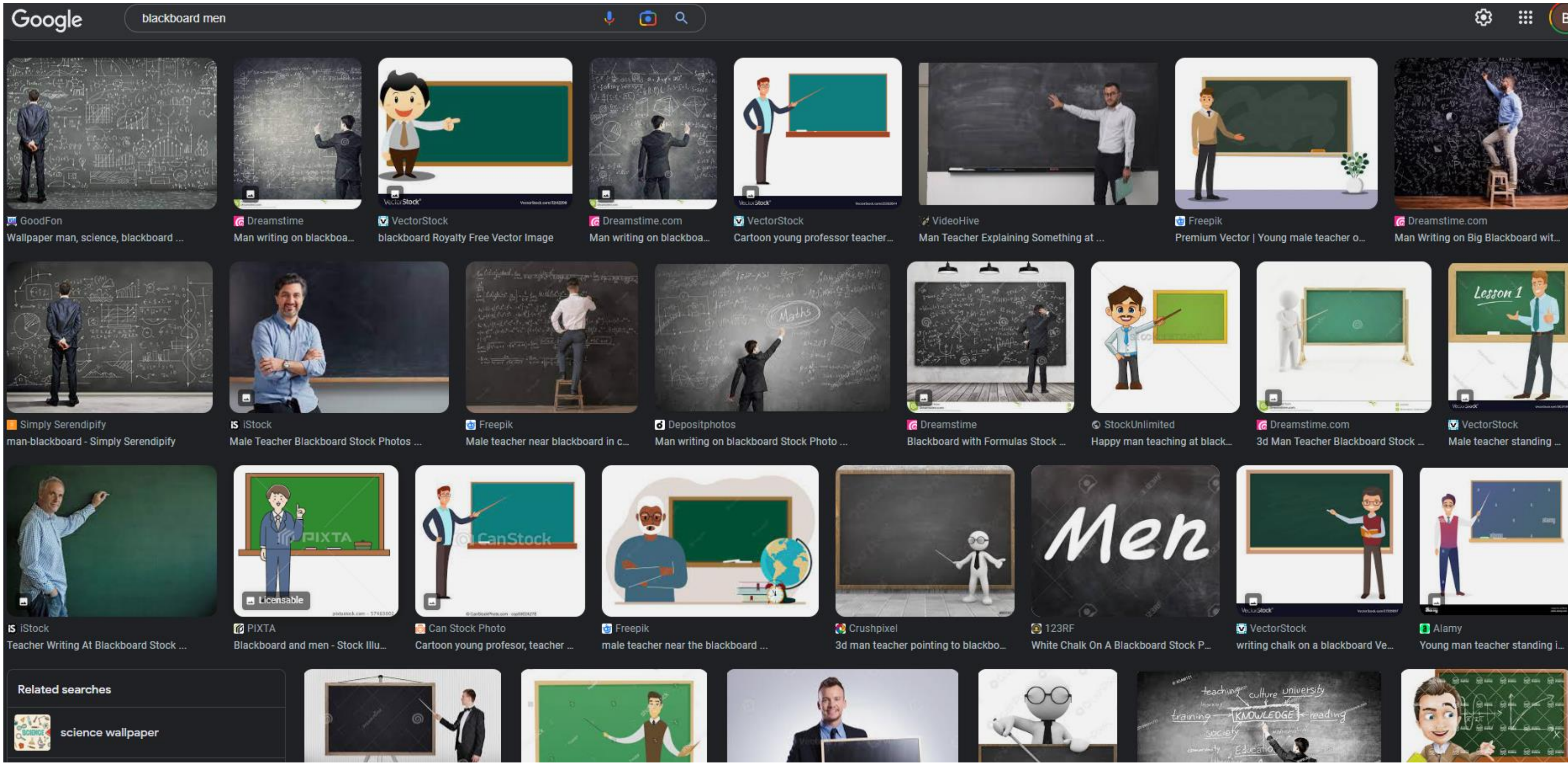
Kerr perturbation theory

- Background: Kerr
- Parameter: often perturbing mass m
- Application: EMRIs, QNMs

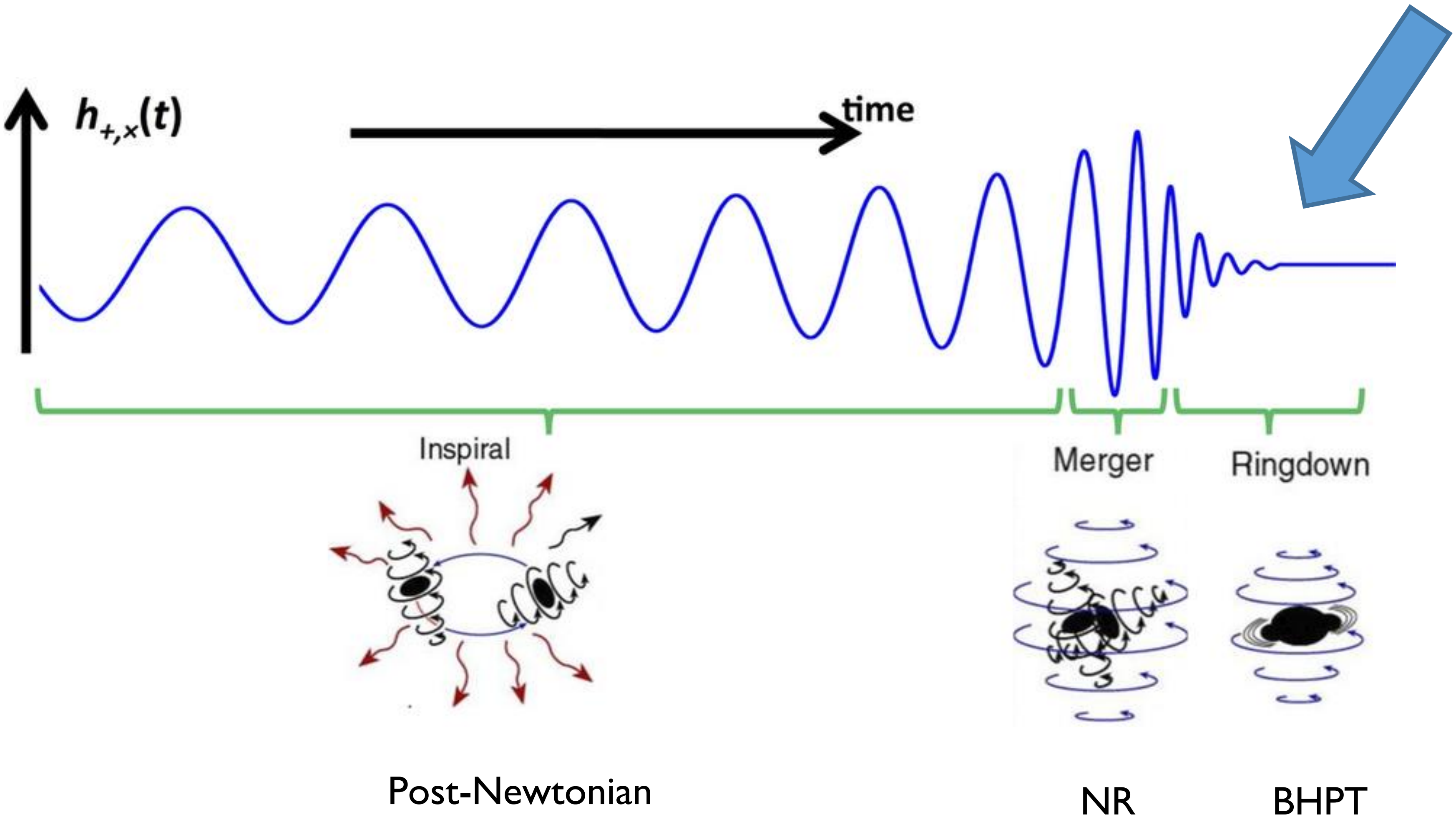
Random lesson: don’t Google blackboard+women



Blackboard+men



Gravitational waves from black hole mergers



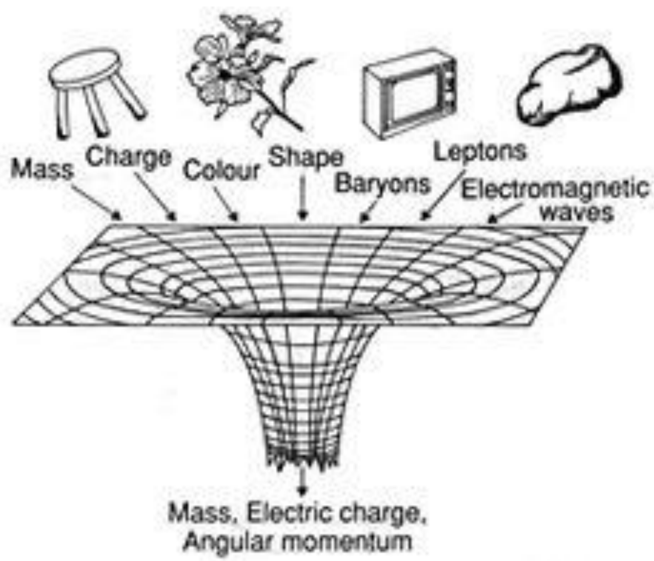
Quasi-normal modes

Black holes are the simplest bells

QNMs depend only on the Mass and Spin.

No hair theorem

"No matter how black hole is formed or what you are throwing in, the resulting spacetime only depends on two parameters!"



How to compute these frequencies?

1. Split your metric $g_{\mu\nu} = g_{\mu\nu}^{Kerr} + \epsilon h_{\mu\nu}$

2. Substitute in Einstein's equations and only keep terms linear in ϵ

3. Use clever variables and gauge choices to write the equation as a ***time-independent Schrodinger equation***

4. Solve for the frequency

Scalar field on Schwarzschild

$$g_{\mu\nu}dx^\mu dx^\nu = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2(d\theta^2 + \sin^2\theta \; d\phi^2)$$

Start with the wave equation

$$\nabla_\mu \nabla^\mu \varphi = \frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}\, g^{\mu\nu}\partial_\nu \varphi) = 0$$

$$\frac{1}{r^2 \sin\theta}\partial_t\left(-\frac{r^2 \sin\theta}{f}\partial_t\varphi\right) + \frac{1}{r^2\sin\theta}\partial_r(r^2 \sin\theta f \partial_r\varphi)$$

$$+\frac{1}{r^2}D^2\varphi = 0$$

Decompose into spherical harmonics

$$\varphi(t,r,\theta,\phi) = \frac{1}{r}\sum_{l,m}\varphi_{lm}(t,r)Y_{lm}(\theta,\phi)$$

Substitute and multiply by $r\,f(r)$

$$-\partial_t^2\varphi_{lm} + f\partial_r(f\partial_r\varphi_{lm}) - \underbrace{f(r)\left(\frac{l(l+1)}{r^2} + \frac{\partial_r f}{r}\right)}_{= V_l^{scalar}}\varphi_{lm} = 0$$

r versus $r_\star$

$$g_{\mu\nu}dx^\mu dx^\nu = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2(d\theta^2 + \sin^2\theta\ d\phi^2)$$
$$= f(-dt^2 + dr_\star^2) + r^2(d\theta^2 + \sin^2\theta\ d\phi^2)$$

$$dr_\star = \frac{dr}{1-\frac{2M}{r}}$$

$$r_\star = r + 2M\ln(\frac{r}{2M} - 1)$$

Simplicity in terms of tortoise coordinate

Note that $f\partial_r(f\partial_r\varphi_{lm}) = \partial_{r_\star}^2\varphi_{lm}$

so that we obtain

$$-\partial_t^2\varphi_{lm} + \partial_{r_\star}^2\varphi_{lm} - V_l(r)\varphi_{lm} = 0$$

Cost: two different radii

Frequency space

$$-\partial_t^2 \varphi_{lm} + \partial_{r_\star}^2 \varphi_{lm} - V_l(r)\varphi_{lm} = 0$$

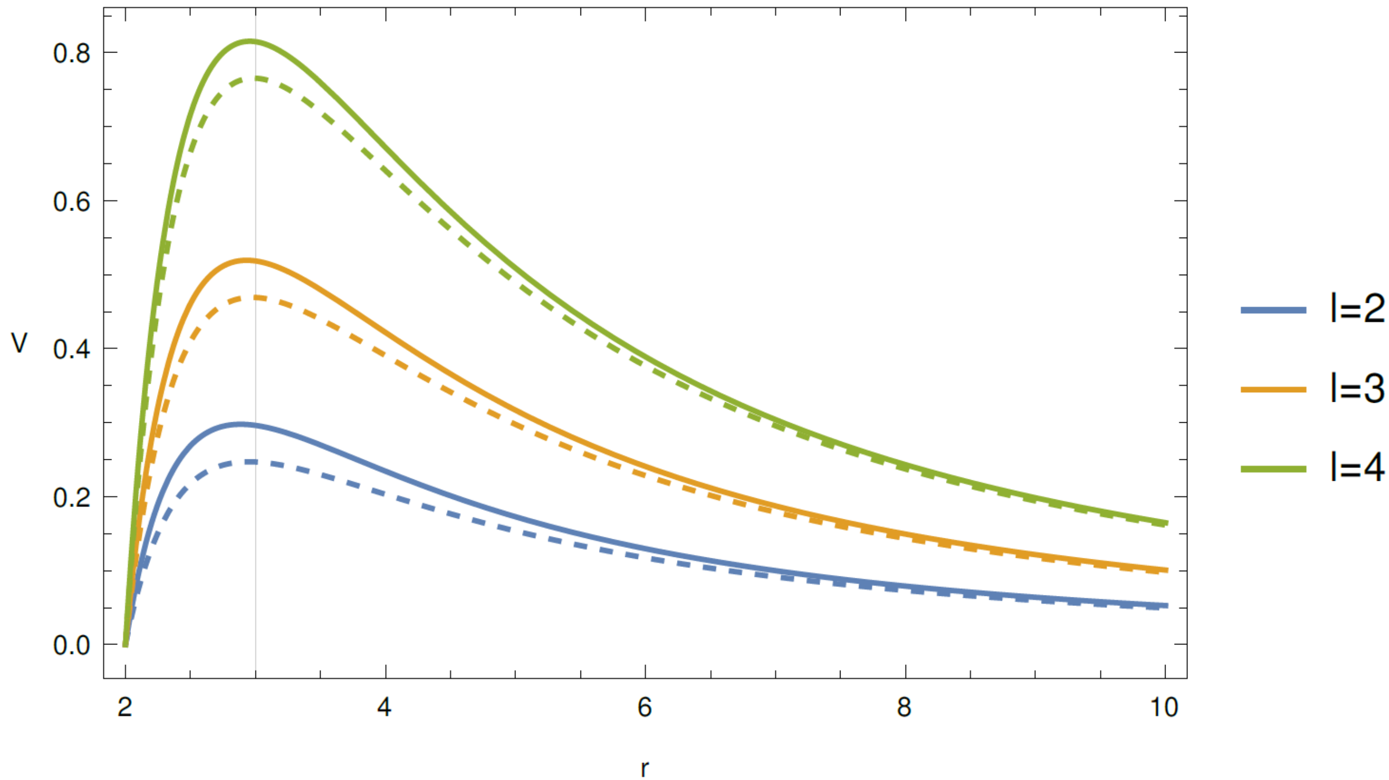
Go to Fourier space

$$\varphi_{lm}(t,r) = \int \frac{d\omega}{\sqrt{2\pi}}\ \tilde{\varphi}_{lm}(\omega, r)e^{i\omega t}$$

to find a time-independent Schrödinger equation

$$\partial_{r_\star}^2 \tilde{\varphi}_{lm} + (\omega_{lm}^2 - V_l(r))\tilde{\varphi}_{lm} = 0$$

Potentials for scalar and gravitational case



$$V_l^{scalar}(r) = f(r)\left(\frac{l(l+1)}{r^2} + \frac{2M}{r}\right)$$

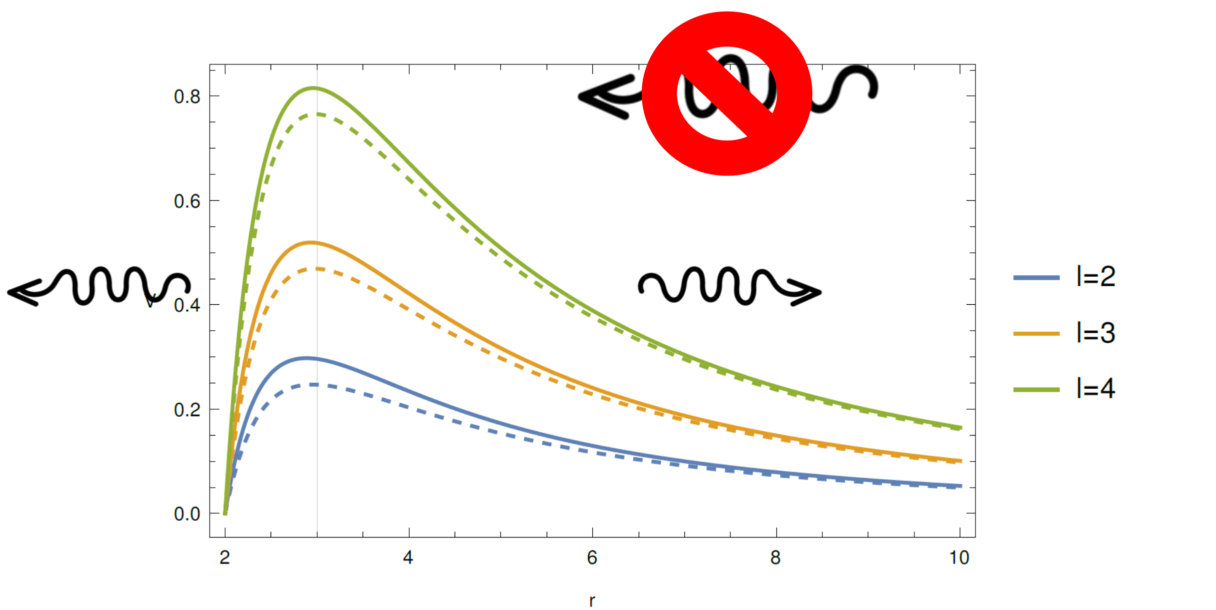
$$V_l^{odd}(r) = f(r)\left(\frac{l(l+1)}{r^2} + \frac{6M}{r}\right)$$

Boundary conditions: "free" oscillation

$$\partial^2_{r_\star}\tilde{\varphi}_{lm} + (\omega^2_{lm} - V_l(r))\tilde{\varphi}_{lm} = 0$$

QNMs are solutions with

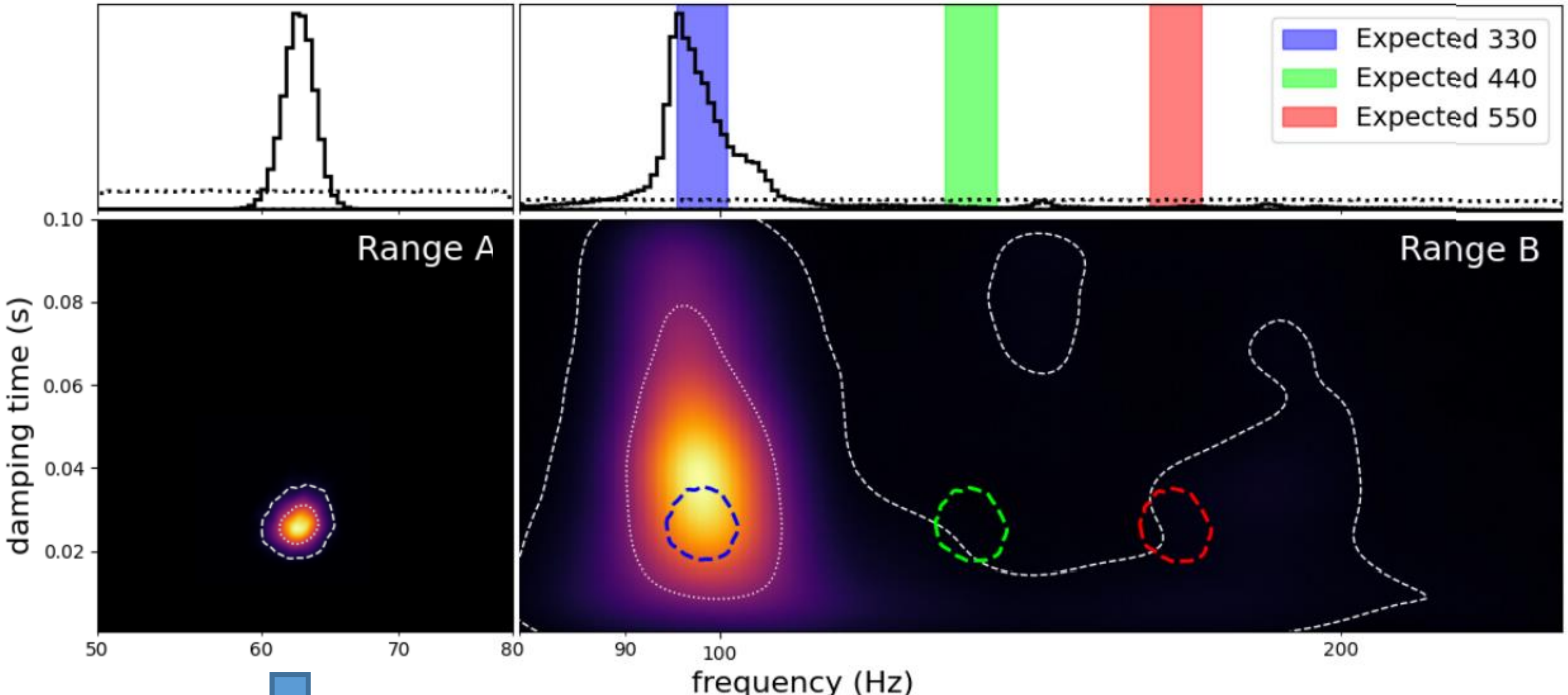
$$\tilde{\varphi}_{lm} = \begin{cases} A(\omega)e^{i\omega r_\star} + A_r(\omega)e^{-i\omega r_\star} & r_\star \to \infty \\ A_t(\omega)e^{i\omega r_\star} & r_\star \to -\infty \end{cases}$$



Can we observe QNMs?

Black hole spectroscopy

GW190521: a massive asymmetric merger

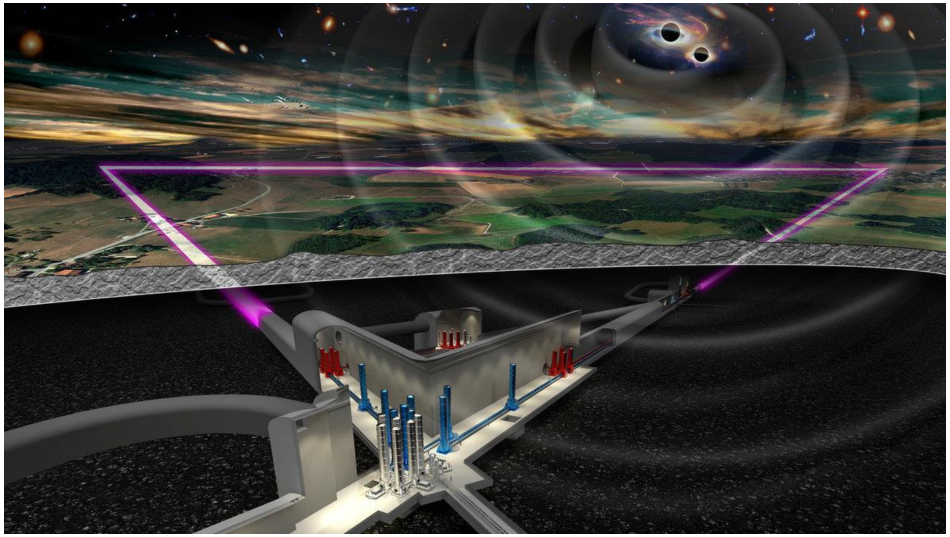


Assume fundamental mode:

Mass and spin BH

The future

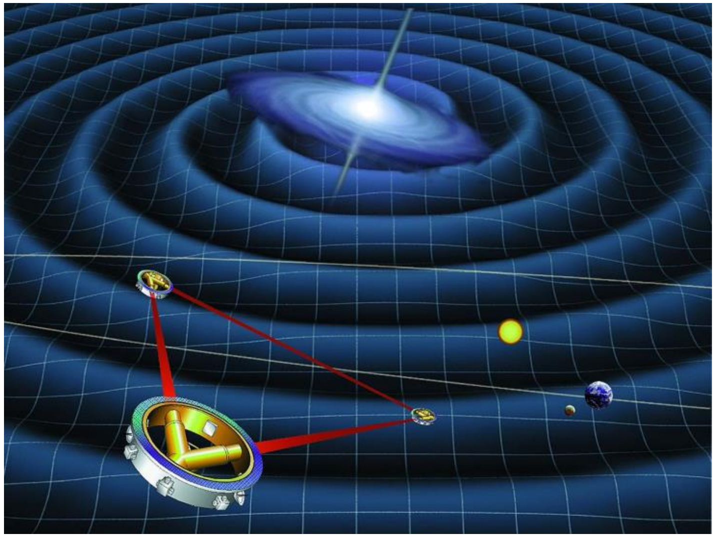
$1 - 10^4$ Hz



Einstein Telescope

(or Cosmic Explorer)

$10^{-4} - 1$ Hz

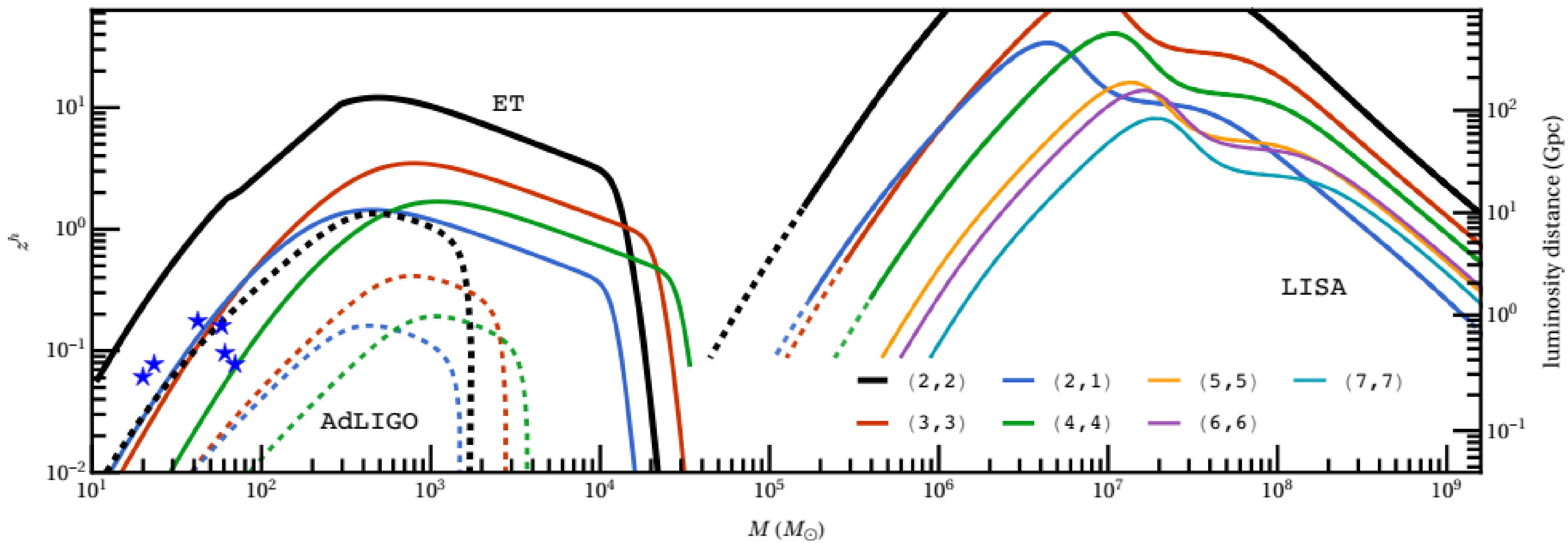


LISA

(or TianQin)

The future

Forecast for non-spinning binary with $m_1 = 2m_2$

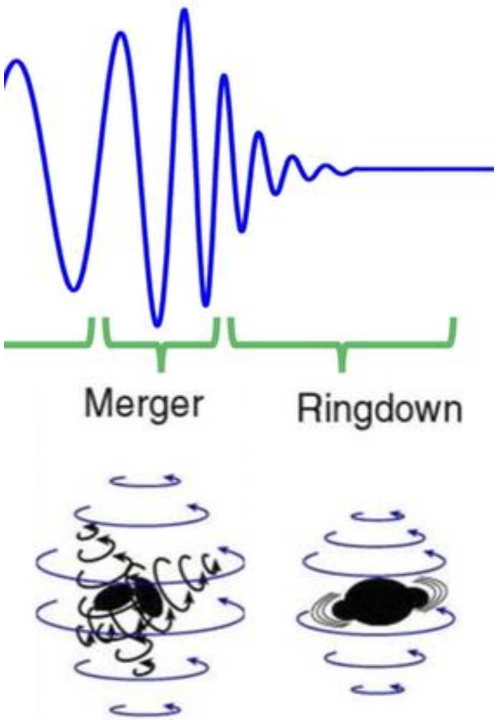


Quasi normal modes are…

1. Awesome
2. “Free” oscillations of a black hole
3. Completely determined by the mass and spin of the final black hole
4. Observable
5. New ways to test the nature of black holes and General Relativity

Non-linearities

When is QNM description valid?

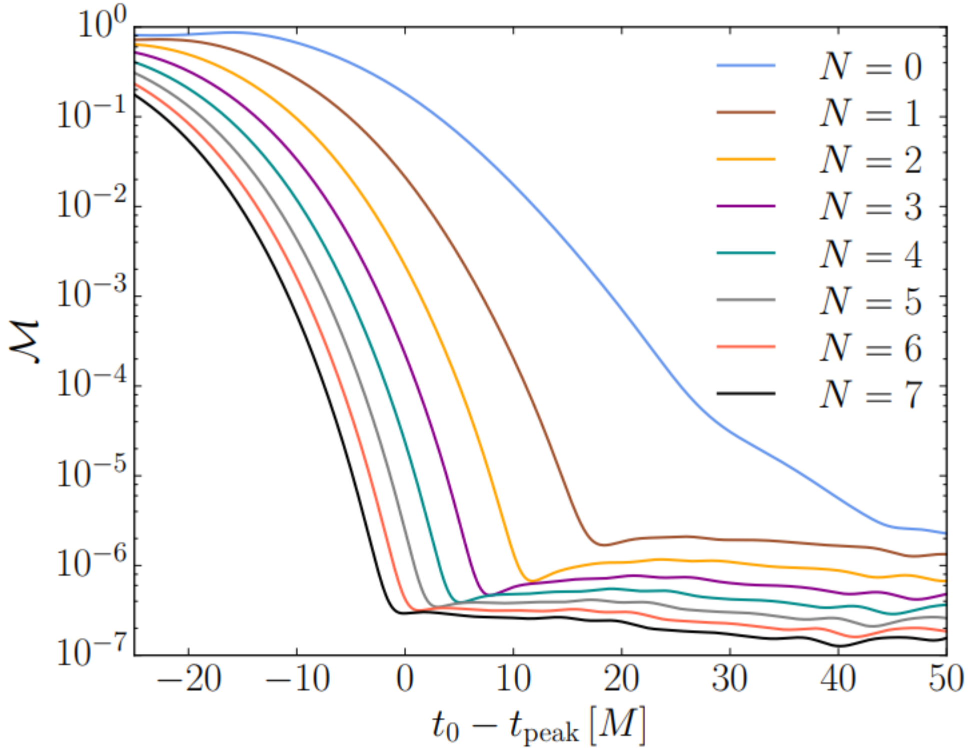


Ringdown starts when…

- $h_+/h_\times$ has a peak?
- Ψ_4 has a peak?
- there is a common (apparent) horizon?
- $10M$ after any of these?
- or $20M$?
- ….?

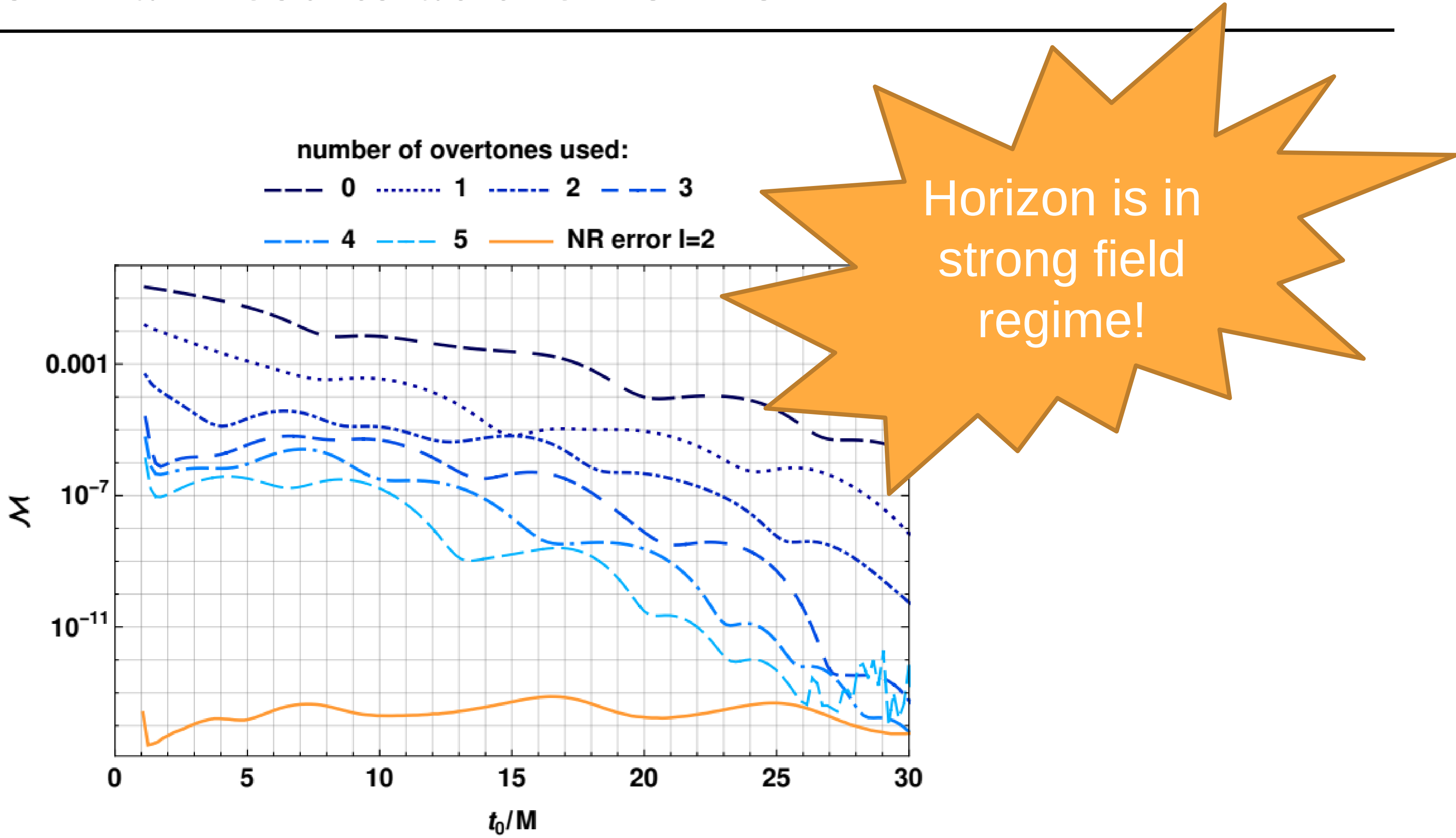
At the peak already?

Observation: many overtones → lower mismatch



[Giesler et al, 2019]

Similar results at the horizon



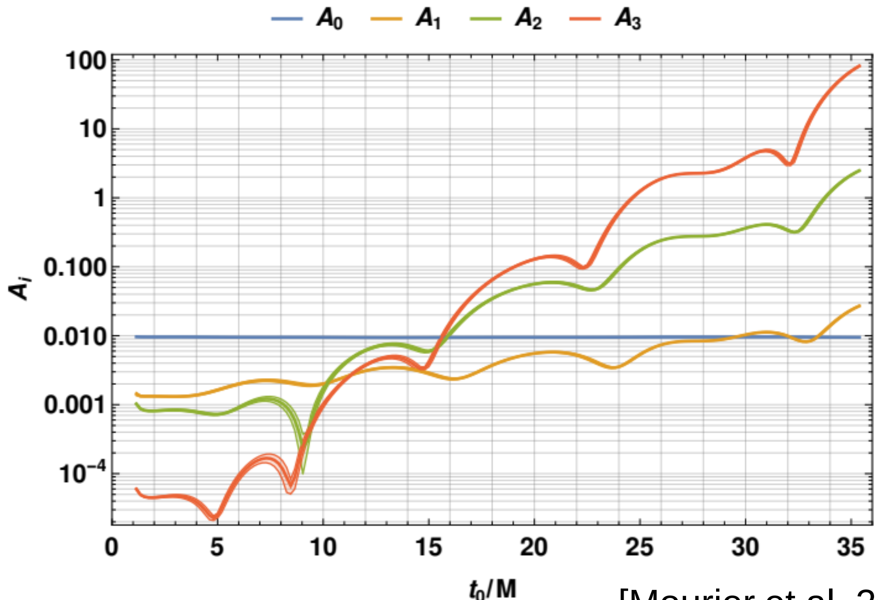
[Mourier et al, 2020]

Conclusion: matter of overfitting

$$h_{lm}(t,r) = \frac{1}{r}\sum_{n=0}^{N} A_{lmn} e^{-i\omega_{lmn}(t-t_0)+\Phi_{lmn}}$$

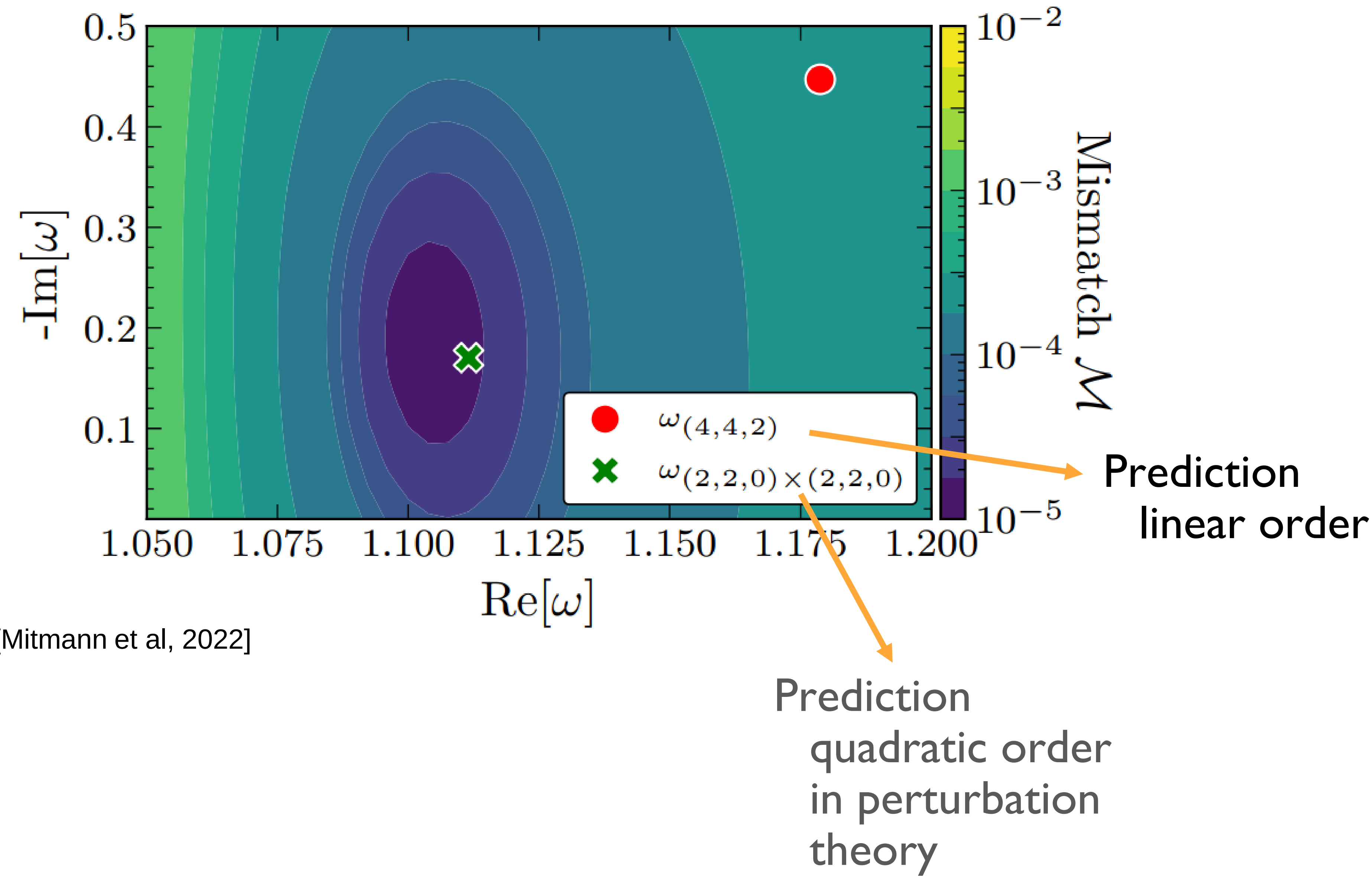
By construction = constants,

but fits show..



[Mourier et al, 2020]

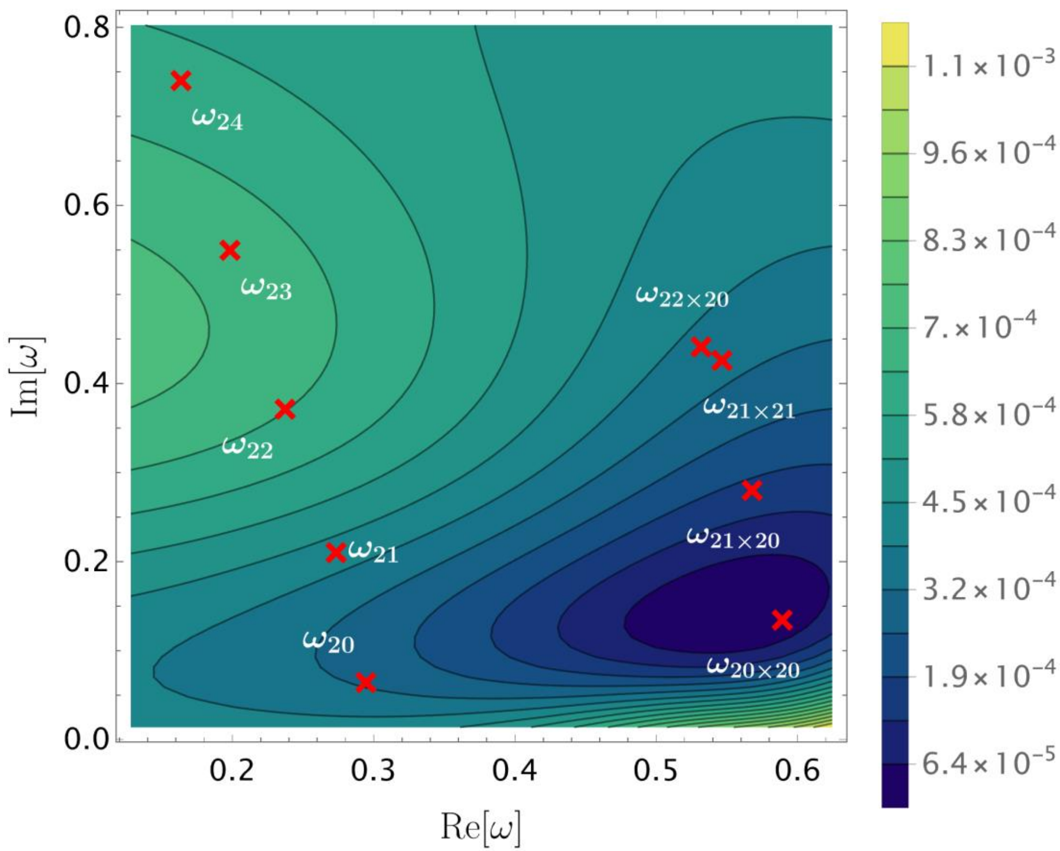
Non-linear model preferred @ infinity



[Mitmann et al, 2022]

Also true @ black hole horizon

Numerical simulation
of head on collision
two black holes

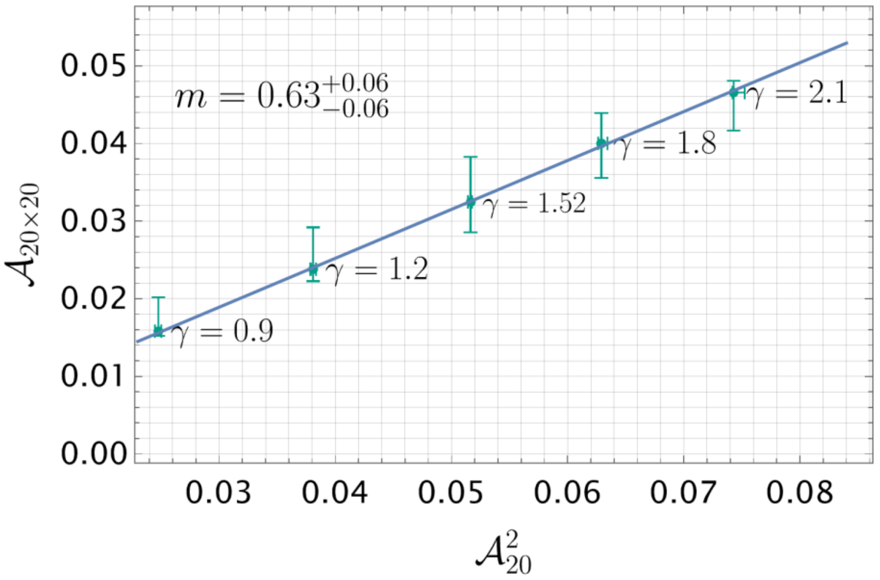


But surprisingly difficult to find!

$$A_{l_1 m_1 \times l_2 m_2} = \alpha\, A_{l_1 m_1} A_{l_2 m_2}$$

Two strategies

1. Use boosted black holes to enhance the linear signal
2. Use very accurate numerical simulations



Take home message

Enjoy the rest of this school!

…and remember quasi-normal modes are awesome (and if you want to know more: ask Ariadna)

Back up slides

Mathematical description

$$h_+ + i\,h_\times = \sum_{l=2}^{\infty}\sum_{m=-l}^{l} h_{lm}(t,r)\ \ {}_2Y_{lm}(\theta,\phi)$$

spin-weighted
spherical harmonic

$$h_{lm}(t,r) = \frac{1}{r}\sum_{n=0}^{N} A_{lmn}e^{-i\omega_{lmn}(t-t_0)+\Phi_{lmn}}$$

Depend on the details
of the "hammer"

Frequencies and damping times

$$\omega_{lmn} = \omega_{lmn}^{R} + i\,\omega_{lmn}^{I} = 2\pi f_{lmn} + \frac{i}{\tau_{lmn}}$$

Depends on three integers:

$$l = 2,3,\ldots$$
$$-l < m < l$$
$$n = 0,1,2,\ldots$$

Damping time

Normal modes: no dissipation

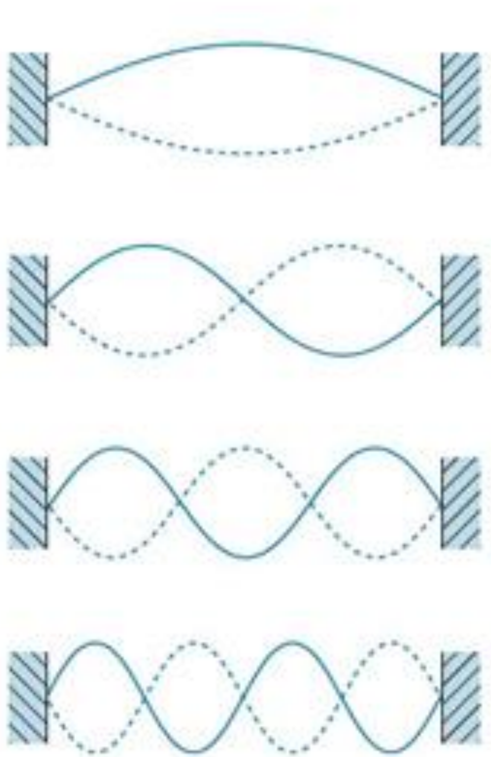
String

$n = 0$

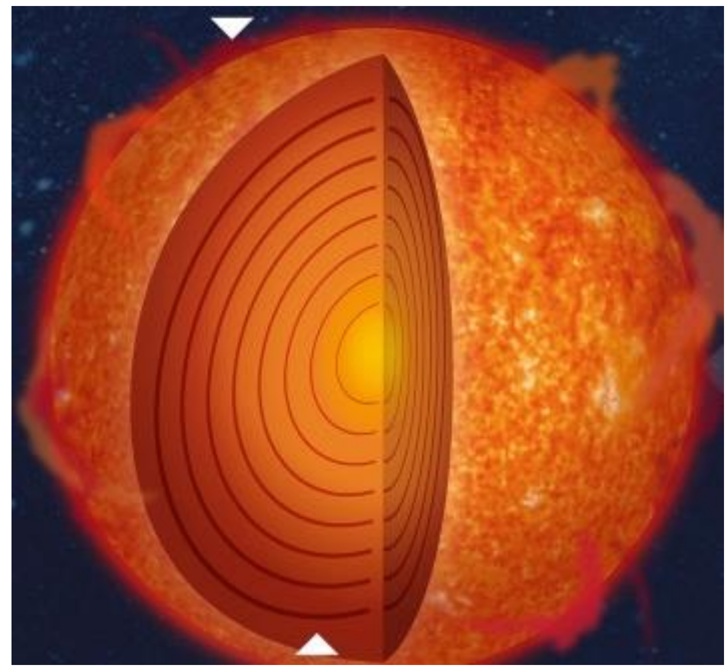
$n = 1$

$n = 2$

$n = 3$



Stars



asteroseismology